George Pólya was born in Hungary in 1887. He received his Ph.D. at the University of Budapest. In 1940 he came to Brown University and then joined the faculty at Stanford University in 1942.

In his studies, he became interested in the process of discovery, which led to his famous four-step process for solving problems:

1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Look back.

Pólya wrote over 250 mathematical papers and three books that promote problem solving. His most famous book, *How to Solve It*, which has been translated into 15 languages, introduced his four-step approach together with heuristics, or strategies, which are helpful in solving problems. Other important works by Pólya are *Mathematical Discovery*, Volumes 1 and 2, and *Mathematics and Plausible Reasoning*, Volumes 1 and 2.

He died in 1985, leaving mathematics with the important legacy of teaching problem solving. His “Ten Commandments for Teachers” are as follows:

1. Be interested in your subject.
2. Know your subject.
3. Try to read the faces of your students; try to see their expectations and difficulties; put yourself in their place.
4. Realize that the best way to learn anything is to discover it by yourself.
5. Give your students not only information, but also know-how, mental attitudes, the habit of methodical work.
6. Let them learn guessing.
7. Let them learn proving.
8. Look out for such features of the problem at hand as may be useful in solving the problems to come—try to disclose the general pattern that lies behind the present concrete situation.
9. Do not give away your whole secret at once—let the students guess before you tell it—let them find out by themselves as much as is feasible.
10. Suggest; do not force information down their throats.
Because problem solving is the main goal of mathematics, this chapter introduces the six strategies listed in the Problem-Solving Strategies box that are helpful in solving problems. Then, at the beginning of each chapter, an initial problem is posed that can be solved by using the strategy introduced in that chapter. As you move through this book, the Problem-Solving Strategies boxes at the beginning of each chapter expand, as should your ability to solve problems.

**Problem-Solving Strategies**
1. Guess and Test
2. Draw a Picture
3. Use a Variable
4. Look for a Pattern
5. Make a List
6. Solve a Simpler Problem

**INITIAL PROBLEM**

Place the whole numbers 1 through 9 in the circles in the accompanying triangle so that the sum of the numbers on each side is 17.

A solution to this Initial Problem is on page 39.
INTRODUCTION

Once, at an informal meeting, a social scientist asked of a mathematics professor, “What’s the main goal of teaching mathematics?” The reply was, “Problem solving.” In return, the mathematician asked, “What is the main goal of teaching the social sciences?” Once more the answer was “Problem solving.” All successful engineers, scientists, social scientists, lawyers, accountants, doctors, business managers, and so on have to be good problem solvers. Although the problems that people encounter may be very diverse, there are common elements and an underlying structure that can help to facilitate problem solving. Because of the universal importance of problem solving, the main professional group in mathematics education, the National Council of Teachers of Mathematics (NCTM), recommended in its 1980 An Agenda for Action that “problem solving be the focus of school mathematics in the 1980s.” The National Council of Teachers of Mathematics’ 1989 Curriculum and Evaluation Standards for School Mathematics called for increased attention to the teaching of problem solving in K–8 mathematics. Areas of emphasis include word problems, applications, patterns and relationships, open-ended problems, and problem situations represented verbally, numerically, graphically, geometrically, or symbolically. The NCTM’s 2000 Principles and Standards for School Mathematics identified problem solving as one of the processes by which all mathematics should be taught.

This chapter introduces a problem-solving process together with six strategies that will aid you in solving problems.

Key Concepts from NCTM Curriculum Focal Points

- **KINDERGARTEN**: Choose, combine, and apply effective strategies for answering quantitative questions.
- **GRADE 1**: Develop an understanding of the meanings of addition and subtraction and strategies to solve such arithmetic problems. Solve problems involving the relative sizes of whole numbers.
- **GRADE 3**: Apply increasingly sophisticated strategies . . . to solve multiplication and division problems.
- **GRADES 4 AND 5**: Select appropriate units, strategies, and tools for solving problems.
- **GRADE 6**: Solve a wide variety of problems involving ratios and rates.
- **GRADE 7**: Use ratio and proportionality to solve a wide variety of percent problems.

### 1.1 THE PROBLEM-SOLVING PROCESS AND STRATEGIES

Use any strategy you know to solve the problem below. As you solve the problem below, pay close attention to the thought processes and steps that you use. Write down these strategies and compare them to a classmate’s. Are there any similarities in your approaches to solving the problem below?

**Problem:** Lin’s garden has an area of 78 square yards. The length of the garden is 5 less than three times its width. What are the dimensions of Lin’s garden?
Pólya’s Four Steps

In this book we often distinguish between “exercises” and “problems.” Unfortunately, the distinction cannot be made precise. To solve an exercise, one applies a routine procedure to arrive at an answer. To solve a problem, one has to pause, reflect, and perhaps take some original step never taken before to arrive at a solution. This need for some sort of creative step on the solver’s part, however minor, is what distinguishes a problem from an exercise. To a young child, finding $3 + 2$ might be a problem, whereas it is a fact for you. For a child in the early grades, the question “How do you divide 96 pencils equally among 16 children?” might pose a problem, but for you it suggests the exercise “find $96 \div 16$.” These two examples illustrate how the distinction between an exercise and a problem can vary, since it depends on the state of mind of the person who is to solve it.

Doing exercises is a very valuable aid in learning mathematics. Exercises help you to learn concepts, properties, procedures, and so on, which you can then apply when solving problems. This chapter provides an introduction to the process of problem solving. The techniques that you learn in this chapter should help you to become a better problem solver and should show you how to help others develop their problem-solving skills.

A famous mathematician, George Pólya, devoted much of his teaching to helping students become better problem solvers. His major contribution is what has become known as Pólya’s four-step process for solving problems.

**Step 1 Understand the Problem**
- Do you understand all the words?
- Can you restate the problem in your own words?
- Do you know what is given?
- Do you know what the goal is?
- Is there enough information?
- Is there extraneous information?
- Is this problem similar to another problem you have solved?

**Step 2 Devise a Plan**
Can one of the following strategies (heuristics) be used? (A strategy is defined as an artful means to an end.)

2. Draw a picture.
3. Use a variable.
4. Look for a pattern.
5. Make a list.
7. Draw a diagram.
8. Use direct reasoning.
9. Use indirect reasoning.
10. Use properties of numbers.
12. Work backward.
13. Use cases.
14. Solve an equation.
15. Look for a formula.
16. Do a simulation.
17. Use a model.
18. Use dimensional analysis.
19. Identify subgoals.
20. Use coordinates.
21. Use symmetry.

The first six strategies are discussed in this chapter; the others are introduced in subsequent chapters.
Section 1.1  The Problem-Solving Process and Strategies

Step 3  Carry Out the Plan

- Implement the strategy or strategies that you have chosen until the problem is solved or until a new course of action is suggested.
- Give yourself a reasonable amount of time in which to solve the problem. If you are not successful, seek hints from others or put the problem aside for a while. (You may have a flash of insight when you least expect it!)
- Do not be afraid of starting over. Often, a fresh start and a new strategy will lead to success.

Step 4  Look Back

- Is your solution correct? Does your answer satisfy the statement of the problem?
- Can you see an easier solution?
- Can you see how you can extend your solution to a more general case?

Usually, a problem is stated in words, either orally or written. Then, to solve the problem, one translates the words into an equivalent problem using mathematical symbols, solves this equivalent problem, and then interprets the answer. This process is summarized in Figure 1.1.

Reflection from Research

Researchers suggest that teachers think aloud when solving problems for the first time in front of the class. In so doing, teachers will be modeling successful problem-solving behaviors for their students (Schoenfeld, 1985).

NCTM Standard

Instructional programs should enable all students to apply and adapt a variety of appropriate strategies to solve problems.

Learning to utilize Pólya’s four steps and the diagram in Figure 1.1 are first steps in becoming a good problem solver. In particular, the “Devise a Plan” step is very important. In this chapter and throughout the book, you will learn the strategies listed under the “Devise a Plan” step, which in turn help you decide how to proceed to solve problems. However, selecting an appropriate strategy is critical! As we worked with students who were successful problem solvers, we asked them to share “clues” that they observed in statements of problems that helped them select appropriate strategies. Their clues are listed after each corresponding strategy. Thus, in addition to learning how to use the various strategies herein, these clues can help you decide when to select an appropriate strategy or combination of strategies. Problem solving is as much an art as it is a science. Therefore, you will find that with experience you will develop a feeling for when to use one strategy over another by recognizing certain clues, perhaps subconsciously. Also, you will find that some problems may be solved in several ways using different strategies.

In summary, this initial material on problem solving is a foundation for your success in problem solving. Review this material on Pólya’s four steps as well as the strategies and clues as you continue to develop your expertise in solving problems.
Problem-Solving Strategies

The remainder of this chapter is devoted to introducing several problem-solving strategies.

**Strategy 1** Guess and Test

**Problem**

Place the digits 1, 2, 3, 4, 5, 6 in the circles in Figure 1.2 so that the sum of the three numbers on each side of the triangle is 12.

We will solve the problem in three ways to illustrate three different approaches to the Guess and Test strategy. As its name suggests, to use the Guess and Test strategy, you guess at a solution and test whether you are correct. If you are incorrect, you refine your guess and test again. This process is repeated until you obtain a solution.

**Step 1** Understand the Problem

Each number must be used exactly one time when arranging the numbers in the triangle. The sum of the three numbers on each side must be 12.

**First Approach: Random Guess and Test**

**Step 2** Devise a Plan

Tear off six pieces of paper and mark the numbers 1 through 6 on them and then try combinations until one works.

**Step 3** Carry Out the Plan

Arrange the pieces of paper in the shape of an equilateral triangle and check sums. Keep rearranging until three sums of 12 are found.

**Second Approach: Systematic Guess and Test**

**Step 2** Devise a Plan

Rather than randomly moving the numbers around, begin by placing the smallest numbers—namely, 1, 2, 3—in the corners. If that does not work, try increasing the numbers to 1, 2, 4, and so on.

**Step 3** Carry Out the Plan

With 1, 2, 3 in the corners, the side sums are too small; similarly with 1, 2, 4. Try 1, 2, 5 and 1, 2, 6. The side sums are still too small. Next try 2, 3, 4, then 2, 3, 5, and so on, until a solution is found. One also could begin with 4, 5, 6 in the corners, then try 3, 4, 5, and so on.

**Third Approach: Inferential Guess and Test**

**Step 2** Devise a Plan

Start by assuming that 1 must be in a corner and explore the consequences.

**Step 3** Carry Out the Plan

If 1 is placed in a corner, we must find two pairs out of the remaining five numbers whose sum is 11 (Figure 1.3). However, out of 2, 3, 4, 5, and 6, only 6 + 5 = 11. Thus, we conclude that 1 cannot be in a corner. If 2 is in a corner, there must be two pairs left that add to 10 (Figure 1.4). But only 6 + 4 = 10. Therefore, 2
cannot be in a corner. Finally, suppose that 3 is in a corner. Then we must satisfy Figure 1.5. However, only \( 5 + 4 = 9 \) of the remaining numbers. Thus, if there is a solution, 4, 5, and 6 will have to be in the corners (Figure 1.6). By placing 1 between 5 and 6, 2 between 4 and 6, and 3 between 4 and 5, we have a solution.

**Step 4** Look Back

Notice how we have solved this problem in three different ways using Guess and Test. Random Guess and Test is often used to get started, but it is easy to lose track of the various trials. Systematic Guess and Test is better because you develop a scheme to ensure that you have tested all possibilities. Generally, Inferential Guess and Test is superior to both of the previous methods because it usually saves time and provides more information regarding possible solutions.

**Additional Problems Where the Strategy “Guess and Test” Is Useful**

1. In the following cryptarithm—that is, a collection of words where the letters represent numbers—sun and fun represent two three-digit numbers, and swim is their four-digit sum. Using all of the digits 0, 1, 2, 3, 6, 7, and 9 in place of the letters where no letter represents two different digits, determine the value of each letter.

\[
\begin{align*}
\text{sun} & + \text{fun} \\
\text{swim} &
\end{align*}
\]

**Step 1** Understand the Problem

Each of the letters in sun, fun, and swim must be replaced with the numbers 0, 1, 2, 3, 6, 7, and 9, so that a correct sum results after each letter is replaced with its associated digit. When the letter \( n \) is replaced by one of the digits, then \( n + n \) must be \( m \) or \( 10 + m \), where the 1 in the 10 is carried to the tens column. Since \( 1 + 1 = 2 \), \( 3 + 3 = 6 \), and \( 6 + 6 = 12 \), there are three possibilities for \( n \), namely, 1, 3, or 6. Now we can try various combinations in an attempt to obtain the correct sum.

**Step 2** Devise a Plan

Use Inferential Guess and Test. There are three choices for \( n \). Observe that sun and fun are three-digit numbers and that swim is a four-digit number. Thus we have to carry when we add \( s \) and \( f \). Therefore, the value for \( s \) in swim is 1. This limits the choices of \( n \) to 3 or 6.

**Step 3** Carry Out the Plan

Since \( s = 1 \) and \( s + f \) leads to a two-digit number, \( f \) must be 9. Thus there are two possibilities:

\[
\begin{align*}
(a) & \quad 1u3 & + 9u3 & \quad \text{1 w i 6} \\
(b) & \quad 1u6 & + 9u6 & \quad \text{1 w i 2}
\end{align*}
\]

In (a), if \( u = 0, 2, \text{ or } 7 \), there is no value possible for \( i \) among the remaining digits. In (b), if \( u = 3 \), then \( u + u \) plus the carry from \( 6 + 6 \) yields \( i = 7 \). This leaves \( w = 0 \) for a solution.
Step 4  Look Back
The reasoning used here shows that there is one and only one solution to this problem. When solving problems of this type, one could randomly substitute digits until a solution is found. However, Inferential Guess and Test simplifies the solution process by looking for unique aspects of the problem. Here the natural places to start are $n + n, u + u,$ and the fact that $s + f$ yields a two-digit number.

2. Use four 4s and some of the symbols $+, \times, -, \div, ()$ to give expressions for the whole numbers from 0 through 9: for example, $5 = (4 \times 4 + 4) \div 4.$

3. For each shape in Figure 1.7, make one straight cut so that each of the two pieces of the shape can be rearranged to form a square.

(Note: Answers for these problems are given after the Solution of the Initial Problem near the end of this chapter.)

CLUES
The Guess and Test strategy may be appropriate when

- There is a limited number of possible answers to test.
- You want to gain a better understanding of the problem.
- You have a good idea of what the answer is.
- You can systematically try possible answers.
- Your choices have been narrowed down by the use of other strategies.
- There is no other obvious strategy to try.

Review the preceding three problems to see how these clues may have helped you select the Guess and Test strategy to solve these problems.

Strategy 2  Draw a Picture
Often problems involve physical situations. In these situations, drawing a picture can help you better understand the problem so that you can formulate a plan to solve the problem. As you proceed to solve the following “pizza” problem, see whether you can visualize the solution without looking at any pictures first. Then work through the given solution using pictures to see how helpful they can be.

Problem
Can you cut a pizza into 11 pieces with four straight cuts?

Step 1  Understand the Problem
Do the pieces have to be the same size and shape?

Step 2  Devise a Plan
An obvious beginning would be to draw a picture showing how a pizza is usually cut and to count the pieces. If we do not get 11, we have to try something else (Figure 1.8). Unfortunately, we get only eight pieces this way.

NCTM Standard
All students should describe, extend, and make generalizations about geometric and numeric patterns.
Problem Solving Workshop
Strategy • Draw a Picture

Learn the Strategy
You can draw a picture to help you solve a math problem. You can use simple marks to stand for people and things.
This picture can be used to show a group of 6 ants and a group of 4 ants.

\[ 6 + 4 = 10 \]

Draw a Math Picture
Draw a picture to find the sum.
There are 5 black spiders and 3 green spiders. There are as many red spiders as there are green spiders.

How many spiders are there in all? Explain.

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Introduction to Problem Solving

**Additional Problems Where the Strategy \"Draw a Picture\" Is Useful**

1. A **tetromino** is a shape made up of four squares where the squares must be joined along an entire side (Figure 1.10). How many different tetromino shapes are possible?

**Step 1** Understand the Problem
The solution of this problem is easier if we make a set of pictures of all possible arrangements of four squares of the same size.

**Step 2** Devise a Plan
Let's start with the longest and narrowest configuration and work toward the most compact.

**Step 3** Carry Out the Plan

Four in a row.

Three in a row with one on top of (or below) the end square.
(Note: The upper square can be at either end—these two are considered to be equivalent.)

Three in a row, with one on top of (or below) the center square.

Two in a row, with one above and one below the two.

Two in a row, with two above.

Were you concerned about cutting equal pieces when you started? That is normal. In the context of cutting a pizza, the focus is usually on trying to cut equal pieces rather than the number of pieces. Suppose that circular cuts were allowed. Does it matter whether the pizza is circular or is square? How many pieces can you get with five straight cuts? \(n\) straight cuts?
Reflection from Research

Given the proper experiences, children as young as eight and nine years of age can learn to comfortably use letters to represent unknown values and can operate on representations involving letters and numbers while fully realizing that they did not know the values of the unknowns (Carraher, Schliemann, Brizuela, & Earnest, 2006).

NCTM Standard

All students should represent the idea of a variable as an unknown quantity using a letter or a symbol.

Algebraic Reasoning

In algebra, the letter “x” is most commonly used for a variable. However, any letter (even Greek letters, for example) can be used as a variable.

Section 1.1 The Problem-Solving Process and Strategies

Step 4 Look Back

Many similar problems can be posed using fewer or more squares. The problems become much more complex as the number of squares increases. Also, new problems can be posed using patterns of equilateral triangles.

2. If you have a chain saw with a bar 18 inches long, determine whether a 16-foot log, 8 inches in diameter, can be cut into 4-foot pieces by making only two cuts.

3. It takes 64 cubes to fill a cubical box that has no top. How many cubes are not touching a side or the bottom?

CLUES

The Draw a Picture strategy may be appropriate when

- A physical situation is involved.
- Geometric figures or measurements are involved.
- You want to gain a better understanding of the problem.
- A visual representation of the problem is possible.

Review the preceding three problems to see how these clues may have helped you select the Draw a Picture strategy to solve these problems.

Strategy 3 Use a Variable

Observe how letters were used in place of numbers in the previous “sun + fun = swim” cryptarithm. Letters used in place of numbers are called variables or unknowns. The Use a Variable strategy, which is one of the most useful problem-solving strategies, is used extensively in algebra and in mathematics that involves algebra.

Problem

What is the greatest number that evenly divides the sum of any three consecutive whole numbers?

By trying several examples, you might guess that 3 is the greatest such number. However, it is necessary to use a variable to account for all possible instances of three consecutive numbers.

Step 1 Understand the Problem

The whole numbers are 0, 1, 2, 3, . . . , so that consecutive whole numbers differ by 1. Thus an example of three consecutive whole numbers is the triple 3, 4, and 5. The sum of three consecutive whole numbers has a factor of 3 if 3 multiplied by another whole number produces the given sum. In the example of 3, 4, and 5, the sum is 12 and $3 \times 4$ equals 12. Thus $3 + 4 + 5$ has a factor of 3.

Step 2 Devise a Plan

Since we can use a variable, say $x$, to represent any whole number, we can represent every triple of consecutive whole numbers as follows: $x$, $x + 1$, $x + 2$. Now we can proceed to see whether the sum has a factor of 3.

Step 3 Carry Out the Plan

The sum of $x$, $x + 1$, and $x + 2$ is

$$x + (x + 1) + (x + 2) = 3x + 3 = 3(x + 1).$$
Thus \( x + (x + 1) + (x + 2) \) is three times \( x + 1 \). Therefore, we have shown that the sum of any three consecutive whole numbers has a factor of 3. The case of \( x = 0 \) shows that 3 is the greatest such number.

**Reflection from Research**

When asked to create their own problems, good problem solvers generated problems that were more mathematically complex than those of less successful problem solvers (Silver & Cai, 1996).

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### Additional Problems Where the Strategy “Use a Variable” Is Useful

1. Find the sum of the first 10, 100, and 500 counting numbers.

**Step 1** Understand the Problem

Since counting numbers are the numbers 1, 2, 3, 4, ..., the sum of the first 10 counting numbers would be \( 1 + 2 + 3 + \ldots + 8 + 9 + 10 \). Similarly, the sum of the first 100 counting numbers would be \( 1 + 2 + 3 + \ldots + 98 + 99 + 100 \) and the sum of the first 500 counting numbers would be \( 1 + 2 + 3 + \ldots + 498 + 499 + 500 \).

**Step 2** Devise a Plan

Rather than solve three different problems, the “Use a Variable” strategy can be used to find a general method for computing the sum in all three situations. Thus, the sum of the first \( n \) counting numbers would be expressed as \( 1 + 2 + 3 + \ldots + (n - 2) + (n - 1) + n \). The sum of these numbers can be found by noticing that the first number 1 added to the last number \( n \) is \( \frac{n}{2} \) the same as \( (n - 1) + 2 \) and \( (n - 2) + 3 \). Adding up all such pairs can be done by adding up all of the numbers twice.

**Step 3** Carry Out the Plan

\[
\begin{align*}
1 & + 2 + 3 + \ldots + (n - 2) + (n - 1) + n \\
\frac{(n + 1) + (n + 1) + (n + 1) + \ldots + (n + 1) + (n + 1) + (n + 1)}{(n + 1) + (n + 1) + (n + 1) + \ldots + (n + 1) + (n + 1) + (n + 1)} \\
& = n \cdot (n + 1)
\end{align*}
\]

Since each number was added twice, the desired sum is obtained by dividing \( n \cdot (n + 1) \) by 2 which yields

\[
1 + 2 + 3 + \ldots + (n - 2) + (n - 1) + n = \frac{n \cdot (n + 1)}{2}
\]

The numbers 10, 100, and 500 can now replace the variable \( n \) to find our desired solutions:

\[
\begin{align*}
1 + 2 + 3 + \ldots + 8 + 9 + 10 &= \frac{10 \cdot (10 + 1)}{2} = 55 \\
1 + 2 + 3 + \ldots + 98 + 99 + 100 &= \frac{100 \cdot (101)}{2} = 5050 \\
1 + 2 + 3 + \ldots + 498 + 499 + 500 &= \frac{500 \cdot 501}{2} = 125,250
\end{align*}
\]
NCTM Standard
All students should develop an initial conceptual understanding of different uses of variables.

Section 1.1 The Problem-Solving Process and Strategies

Step 4 Look Back
Since the method for solving this problem is quite unique could it be used to solve other similar looking problems like:

i. \[3 + 6 + 9 + \ldots + (3n - 6) + (3n - 3) + 3n\]

ii. \[21 + 25 + 29 + \ldots + 113 + 117 + 121\]

2. Show that the sum of any five consecutive odd whole numbers has a factor of 5.

3. The measure of the largest angle of a triangle is nine times the measure of the smallest angle. The measure of the third angle is equal to the difference of the largest and the smallest. What are the measures of the angles? (Recall that the sum of the measures of the angles in a triangle is 180°.)

CLUES
The Use a Variable strategy may be appropriate when

- A phrase similar to “for any number” is present or implied.
- A problem suggests an equation.
- A proof or a general solution is required.
- A problem contains phrases such as “consecutive,” “even,” or “odd” whole numbers.
- There is a large number of cases.
- There is an unknown quantity related to known quantities.
- There is an infinite number of numbers involved.
- You are trying to develop a general formula.

Review the preceding three problems to see how these clues may have helped you select the Use a Variable strategy to solve these problems.

Using Algebra to Solve Problems
To effectively employ the Use a Variable strategy, students need to have a clear understanding of what a variable is and how to write and simplify equations containing variables. This subsection addresses these issues in an elementary introduction to algebra. There will be an expanded treatment of solving equations and inequalities in Chapter 9 after the real number system has been developed.

A common way to introduce the use of variables is to find a general formula for a pattern of numbers such as 3, 6, 9, \ldots 3n. One of the challenges for students is to see the role that each number plays in the expression. For example, the pattern 5, 8, 11, \ldots is similar to the previous pattern, but it is more difficult to see that each term is two greater than a multiple of 3 and, thus, can be expressed in general as 3n + 2. Sometimes it is easier for students to use a variable to generalize a geometric pattern such as the one shown in the following example. This type of example may be used to introduce seventh-grade students to the concept of a variable. Following are four typical student solutions.
Describe at least four different ways to count the dots in Figure 1.11.

**SOLUTION**

The obvious method of solution is to count the dots—there are 16. Another student’s method is illustrated in Figure 1.12.

\[
4 \times 3 + 4 \quad \text{and} \quad 4 \times (5 - 2) + 4
\]

The student counts the number of interior dots on each side, 3, and multiplies by the number of sides, 4, and then adds the dots in the corners, 4. This method generates the expression \( 4 \times 3 + 4 = 16 \). A second way to write this expression is \( 4 \times (5 - 2) + 4 = 16 \) since the 3 interior dots can be determined by subtracting the two corners from the 5 dots on a side. Both of these methods are shown in Figure 1.12.

A third method is to count all of the dots on a side, 5, and multiply by the number of sides. Four must then be subtracted because each corner has been counted twice, once for each side it belongs to. This method is illustrated in Figure 1.13 and generates the expression shown.

\[
4 \times 5 - 4 = 16
\]

In the two previous methods, either corner dots are not counted (so they must be added on) or they are counted twice (so they must be subtracted to avoid double counting). The following fourth method assigns each corner to only one side (Figure 1.14).

\[
4 \times 4 = 16 \quad \text{and} \quad 4 \times (5 - 1) = 16
\]

Thus, we encircle 4 dots on each side and multiply by the number of sides. This yields the expression \( 4 \times 4 = 16 \). Because the 4 dots on each side come from the 5 total dots on a side minus 1 corner, this expression could also be written as \( 4 \times (5 - 1) = 16 \) (see Figure 1.14).
There are many different methods for counting the dots in the previous example and each method has a geometric interpretation as well as a corresponding arithmetic expression. Could these methods be generalized to 50, 100, 1000 or even \( n \) dots on a side? The next example discusses how these generalizations can be viewed as well as displays the generalized solutions of seventh-grade students.

**Example 1.2** Suppose the square arrangement of dots in Example 1.1 had \( n \) dots on each side. Write an algebraic expression that would describe the total number of dots in such a figure (Figure 1.15).

**SOLUTION** It is easier to write a general expression for those in Example 1.1 when you understand the origins of the numbers in each expression. In all such cases, there will be 4 corners and 4 sides, so the values that represent corners and sides will stay fixed at 4. On the other hand, in Figure 1.15, any value that was determined based on the number of dots on the side will have to reflect the value of \( n \). Thus, the expressions that represent the total number of dots on a square figure with \( n \) dots on a side are generalized as shown next.

\[
\begin{align*}
4 \times 3 + 4 & \quad \rightarrow 4(n - 2) + 4 \\
4 \times (5 - 2) + 4 & \quad \rightarrow 4n - 4 \\
4 \times 5 - 4 & \quad \rightarrow 4n - 4 \\
4 \times 4 & \quad \rightarrow 4(n - 1)
\end{align*}
\]

Since each expression on the right represents the total number of dots in Figure 1.15, they are all equal to each other. Using properties of numbers and equations, each equation can be rewritten as the same expression. Learning to simplify expressions and equations with variables is one of the most important processes in mathematics. Traditionally, this topic has represented a substantial portion of an entire course in introductory algebra. An equation is a sentence involving numbers, or symbols representing numbers where the verb is equals (=). There are various types of equations:

- **True equation**: \( 3 + 4 = 7 \)
- **False equation**: \( 3 + 4 = 9 \)
- **Identity equation**: \( 2x + 5x = 7x \)
- **Conditional equation**: \( x + 4 = 9 \)

A true or false equation needs no explanation, but an identity equation is always true no matter what numerical value is used for \( x \). A conditional equation is an equation that is only true for certain values of \( x \). For example, the equation \( x + 4 = 9 \) is true when \( x = 5 \), but false when \( x \) is any other value. In this chapter, we will restrict the variables to only whole numbers. For a conditional equation, a value of the variable that makes the equation true is called a solution. To solve an equation means to find all of the solutions. The following example shows three different ways to solve equations of the form \( ax + b = c \).
Example 1.3

Suppose the square arrangement of dots in Example 1.2 had 84 total dots (Figure 1.16). How many dots are there on each side?

Figure 1.16 84 total dots

SOLUTION From Example 1.2, a square with \( n \) dots on a side has \( 4n - 4 \) total dots. Thus, we have the equation \( 4n - 4 = 84 \). Three elementary methods that can be used to solve equations such as \( 4n - 4 = 84 \) are Guess and Test, Cover Up, and Work Backward.

Guess and Test As the name of this method suggests, one guesses values for the variable in the equation \( 4n - 4 = 84 \) and substitutes to see if a true equation results.

Try \( n = 10 \): \( 4(10) - 4 = 36 \neq 84 \)

Try \( n = 25 \): \( 4(25) - 4 = 96 \neq 84 \)

Try \( n = 22 \): \( 4(22) - 4 = 84 \). Therefore, 22 is the solution of the equation.

Cover Up In this method, we cover up the term with the variable:

\[
\square - 4 = 84. \text{ To make a true equation, the } \square \text{ must be 88. Thus } 4n = 88. \text{ Since } 4 \cdot 22 = 88, \text{ we have } n = 22.
\]

Work Backward The left side of the equation shows that \( n \) is multiplied by 4 and then 4 is subtracted to obtain 84. Thus, working backward, if we add 4 to 84 and divide by 4, we reach the value of \( n \). Here \( 84 + 4 = 88 \) and \( 88 \div 4 = 22 \) so \( n = 22 \) (Figure 1.17).

Using variables in equations and manipulating them are what most people see as “algebra.” However, algebra is much more than manipulating variables—it includes the reasoning that underlies those manipulations. In fact, many students solve algebra-like problems without equations and don’t realize that their reasoning is algebraic. For example, the Work Backward solution in Example 1.3 can all be done without really thinking about the variable \( n \). If one just thinks “four times something minus 4 is 84,” he can work backward to find the solution. However, the thinking that is used in the Work Backward method can be mirrored in equations as follows:

\[
4n - 4 = 84 \quad \Rightarrow \quad 4n = 84 + 4 \quad \Rightarrow \quad 4n = 88 \quad \Rightarrow \quad n = 88 \div 4 \quad \Rightarrow \quad n = 22
\]

Reflection from Research
We are proposing that the teaching and learning of arithmetic be conceived as part of the foundation of learning algebra, not that algebra be conceived only as an extension of arithmetic procedures (Carpenter, Levi, Berman, & Pligge, 2005).
Find a set of consecutive counting numbers whose sum is each of the following. Each set may consist of 2, 3, 4, 5, or 6 consecutive integers. Use the spreadsheet activity Consecutive Integer Sum on our Web site to assist you.

a. 84  
b. 213  
c. 154

Section 1.1 The Problem-Solving Process and Strategies

There is a story about Sir Isaac Newton, coinventor of the calculus, who, as a youngster, was sent out to cut a hole in the barn door for the cats to go in and out. With great pride he admitted to cutting two holes, a larger one for the cat and a smaller one for the kittens.

Mathematical Morsel

There is a story about Sir Isaac Newton, coinventor of the calculus, who, as a youngster, was sent out to cut a hole in the barn door for the cats to go in and out. With great pride he admitted to cutting two holes, a larger one for the cat and a smaller one for the kittens.

Section 1.1 Exercise / Problem Set A

1. a. If the diagonals of a square are drawn in, how many triangles of all sizes are formed?  
b. Describe how Pólya's four steps were used to solve part a.

2. Scott and Greg were asked to add two whole numbers. Instead, Scott subtracted the two numbers and got 10, and Greg multiplied them and got 651. What was the correct sum?

3. The distance around a standard tennis court is 228 feet. If the length of the court is 6 feet more than twice the width, find the dimensions of the tennis court.

4. A multiple of 11 I be,  
not odd, but even, you see.  
My digits, a pair,  
when multiplied there,  
make a cube and a square  
out of me. Who am I?

5. Show how 9 can be expressed as the sum of two consecutive numbers. Then decide whether every odd number can be expressed as the sum of two consecutive counting numbers. Explain your reasoning.

6. Using the symbols +, −, ×, and ÷, fill in the following three blanks to make a true equation. (A symbol may be used more than once.)

6  6  6  6 = 13

7. In the accompanying figure (called an arithmogon), the number that appears in a square is the sum of the numbers in the circles on each side of it. Determine what numbers belong in the circles.

8. Place 10 stools along four walls of a room so that each of the four walls has the same number of stools.

9. Susan has 10 pockets and 44 dollar bills. She wants to arrange the money so that there are a different number of dollars in each pocket. Can she do it? Explain.

10. Arrange the numbers 2, 3, . . ., 10 in the accompanying triangle so that each side sums to 21.

11. Find a set of consecutive counting numbers whose sum is each of the following. Each set may consist of 2, 3, 4, 5, or 6 consecutive integers. Use the spreadsheet activity Consecutive Integer Sum on our Web site to assist you.

a. 84  
b. 213  
c. 154
12. Place the digits 1 through 9 so that you can count from 1 to 9 by following the arrows in the diagram.

13. Using a 5-minute and an 8-minute hourglass timer, how can you measure 1 minute?

14. Using the numbers 9, 8, 7, 6, 5, and 4 once each, find the following:
   a. The largest possible sum:
   b. The smallest possible (positive) difference:

15. Using the numbers 1 through 8, place them in the following eight squares so that no two consecutive numbers are in touching squares (touching includes entire sides or simply one point).

16. Solve this cryptarithm, where each letter represents a digit and no digit represents two different letters:

   USSR
   + USA
   PEACE

17. On a balance scale, two spools and one thimble balance eight buttons. Also, one spool balances one thimble and one button. How many buttons will balance one spool?

18. Place the numbers 1 through 8 in the circles on the vertices of the accompanying cube so that the difference of any two connecting circles is greater than 1.


20. The digits 1 through 9 can be used in decreasing order, with + and − signs, to produce 100 as shown: $98 - 76 + 54 + 3 + 21 = 100$. Find two other such combinations that will produce 100.

21. The Indian mathematician Ramanujan observed that the taxi number 1729 was very interesting because it was the smallest counting number that could be expressed as the sum of cubes in two different ways. Find $a$, $b$, $c$, and $d$ such that $a^3 + b^3 = 1729$ and $c^3 + d^3 = 1729$.

22. Using the Chapter 1 eManipulative activity Number Puzzles, Exercise 2 on our Web site, arrange the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 in the following circles so the sum of the numbers along each line of four is 23.

23. Using the Chapter 1 eManipulative activity Circle 21 on our Web site, find an arrangement of the numbers 1 through 14 in the 7 circles below so that the sum of the three numbers in each circle is 21.

24. The hexagon below has a total of 126 dots and an equal number of dots on each side. How many dots are on each side?
EXERCISE / PROBLEM SET B

1. Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1s are separated by one digit, the 2s by two digits, the 3s by three digits, and the 4s by four digits.


3. Carol bought some items at a variety store. All the items were the same price, and she bought as many items as the price of each item in cents. (For example, if the items cost 10 cents, she would have bought 10 of them.) Her bill was $2.25. How many items did Carol buy?

4. You can make one square with four toothpicks. Show how you can make two squares with seven toothpicks (breaking toothpicks is not allowed), three squares with 10 toothpicks, and five squares with 12 toothpicks.

5. A textbook is opened and the product of the page numbers of the two facing pages is 6162. What are the numbers of the pages?

6. Place numbers 1 through 19 into the 19 circles below so that any three numbers in a line through the center will give the same sum.

7. Using three of the symbols +, −, ×, and ÷ once each, fill in the following three blanks to make a true equation. (Parentheses are allowed.)

8. A water main for a street is being laid using a particular kind of pipe that comes in either 18-foot sections or 20-foot sections. The designer has determined that the water main would require 14 fewer sections of 20-foot pipe than if 18-foot sections were used. Find the total length of the water main.

9. Mike said that when he opened his book, the product of the page numbers of the two facing pages was 7007. Without performing any calculations, prove that he was wrong.

10. The Smiths were about to start on an 18,000-mile automobile trip. They had their tires checked and found that each was good for only 12,000 miles. What is the smallest number of spares that they will need to take along with them to make the trip without having to buy a new tire?

11. What is the maximum number of pieces of pizza that can result from 4 straight cuts?

12. Given: Six arrows arranged as follows:

Goal: By inverting two adjacent arrows at a time, rearrange to the following:

Can you find a minimum number of moves?

13. Two friends are shopping together when they encounter a special “3 for 2” shoe sale. If they purchase two pairs of shoes at the regular price, a third pair (of lower or equal value) will be free. Neither friend wants three pairs of shoes, but Pat would like to buy a $56 and a $39 pair while Chris is interested in a $45 pair. If they buy the shoes together to take advantage of the sale, what is the fairest share for each to pay?

14. Find digits A, B, C, and D that solve the following cryptarithm.

15. If possible, find an odd number that can be expressed as the sum of four consecutive counting numbers. If impossible, explain why.

16. Five friends were sitting on one side of a table. Gary sat next to Bill. Mike sat next to Tom. Howard sat in the third seat from Bill. Gary sat in the third seat from Mike. Who sat on the other side of Tom?

17. In the following square array on the left, the corner numbers were given and the boldface numbers were found by adding the adjacent corner numbers. Following the same rules, find the corner numbers for the other square array.

18. Together, a baseball and a football weigh 1.25 pounds, the baseball and a soccer ball weigh 1.35 pounds, and the football and the soccer ball weigh 1.9 pounds. How much does each of the balls weigh?
20 Chapter 1 Introduction to Problem Solving

19. Pick any two consecutive numbers. Add them. Then add 9 to the sum. Divide by 2. Subtract the smaller of the original numbers from the answer. What did you get? Repeat this process with two other consecutive numbers. Make a conjecture (educated guess) about the answer, and prove it.

20. An additive magic square has the same sum in each row, column, and diagonal. Find the error in this magic square and correct it.

\[
\begin{array}{cccc}
47 & 56 & 34 & 22 \\
24 & 67 & 44 & 26 \\
29 & 52 & 3 & 99 \\
17 & 49 & 89 & 4 \\
97 & 6 & 3 & 11 \\
35 & 19 & 46 & 87 \\
\end{array}
\]

21. Two points are placed on the same side of a square. A segment is drawn from each of these points to each of the 2 vertices (corners) on the opposite side of the square. How many triangles of all sizes are formed?

22. Using the triangle in Problem 10 in Part A, determine whether you can make similar triangles using the digits 1, 2, . . . , 9, where the side sums are 18, 19, 20, 21, and 22.

23. Using the Chapter 1 eManipulative activity, Number Puzzles, Exercise 4 on our Web site, arrange the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 in the circles below so the sum of the numbers along each line of four is 20.

24. Using the Chapter 1 eManipulative activity Circle 99 on our Web site, find an arrangement of the numbers provided in the 7 circles below so that the sum of the three numbers in each circle is 99.

25. An arrangement of dots forms the perimeter of an equilateral triangle. There are 87 evenly spaced dots on each side including the dots at the vertices. How many dots are there altogether?

26. The equation \( \frac{y}{5} + 12 = 23 \) can be solved by subtracting 12 from both sides of the equation to yield \( \frac{y}{5} = 11 \). Similarly, the resulting equation \( \frac{y}{5} = 11 \) can be solved by multiplying both sides of the equation by 5 to obtain \( y = 55 \). Explain how this process is related to the Work Backward method described in Example 1.3.

Analyzing Student Thinking

27. When the class was asked to solve the equation \( 3x - 8 = 27 \), Wesley asked if he could use guess and test. How would you respond?

28. Rosemary said that she felt the “Guess and Test” method was a waste of time; she just wanted to get an answer. What could you tell her about the value of using guess and test?

29. Even though you have taught your students how to “draw a picture” to solve a problem, Cecelia asks if she has to draw a picture because she can solve the problems in class without it. How would you respond?

30. When Damian, a second grader, was asked to solve problems like \( x + 3 = 5 \), he said that he had seen his older sister working on problems like \( x + 3 = 5 \) and wondered if these equations were different. How would you respond?

31. After the class had found three consecutive odd numbers whose sum is 99, Byron tried to find three consecutive odd numbers that would add to 96. He said he was struggling to find a solution. How could you help him understand the solution to this problem?

32. Consider the following problem:

The amount of fencing needed to enclose a rectangular field was 92 yards and the length of the field was 3 times as long as the width. What were the dimensions of the field?

Vance solved this problem by drawing a picture and using guess and test. Jolie set up an equation with \( x \) being the width of the field and solved it. They got the same answer but asked you which method was better. How would you respond?
### Problems Related to the NCTM Standards and Curriculum Focal Points

1. The Focal Points for Grades 4 and 5 state “Select appropriate units, strategies, and tools for solving problems.” Find a problem or example in this section that illustrates this statement and explain your reasoning.

2. The NCTM Standards state “All students should describe, extend, and make generalizations about geometric and numeric patterns.” Find a problem in this problem set that illustrates this statement and explain your reasoning.

3. The NCTM Standards state “All students should develop an initial conceptual understanding of different uses of variables.” Find a problem or example in this section that illustrates this statement and explain your reasoning.

### 1.2 THREE ADDITIONAL STRATEGIES

#### STARTING POINT

Solve the problem below using Pólya’s four steps and any other strategy. Describe how you used the four steps, focusing on any new insights that you gained as a result of looking back. How many rectangles of all shapes and sizes are in the figure at the right?

#### Strategy 4  Look for a Pattern

When using the Look for a Pattern strategy, one usually lists several specific instances of a problem and then looks to see whether a pattern emerges that suggests a solution to the entire problem. For example, consider the sums produced by adding consecutive odd numbers starting with 1:

\[ 1, 1 + 3 = 4 (= 2 \times 2), 1 + 3 + 5 = 9 (= 3 \times 3), 1 + 3 + 5 + 7 = 16 (= 4 \times 4), 1 + 3 + 5 + 7 + 9 = 25 (= 5 \times 5), \text{ and so on}. \]

Based on the pattern generated by these five examples, one might expect that such a sum will always be a perfect square.

The justification of this pattern is suggested by the following figure.

Each consecutive odd number of dots can be added to the previous square arrangement to form another square. Thus, the sum of the first \( n \) odd numbers is \( n^2 \).

Generalizing patterns, however, must be done with caution because with a sequence of only 3 or 4 numbers, a case could be made for more than one pattern. For example, consider the sequence 1, 2, 4, . . . . What are the next 4 numbers in the sequence? It can be seen that 1 is doubled to get 2 and 2 is doubled to get 4. Following that pattern, the next four numbers would be 8, 16, 32, 64. If, however, it is noted that the difference between the first and second term is 1 and the difference between the second and third term is 2, then a case could be made that the difference is increasing by one. Thus, the next four terms would be 7, 11, 16, 22. Another case could be made for the differences...
Step 4

alternating between 1 and 2. In that case, the next four terms would be 5, 7, 8, 10. Thus, from the initial three numbers of 1, 2, 4, at least three different patterns are possible:

- 1, 2, 4, 8, 16, 32, 64, \ldots \text{ Doubling}
- 1, 2, 4, 7, 11, 16, 22, \ldots \text{ Difference increasing by 1}
- 1, 2, 4, 5, 7, 8, 10, \ldots \text{ Difference alternating between 1 and 2}

**Problem**

How many different downward paths are there from A to B in the grid in Figure 1.18? A path must travel on the lines.

**Step 1** Understand the Problem

What do we mean by different and downward? Figure 1.19 illustrates two paths. Notice that each such path will be 6 units long. **Different** means that they are not exactly the same; that is, some part or parts are different.

**Step 2** Devise a Plan

Let’s look at each point of intersection in the grid and see how many different ways we can get to each point. Then perhaps we will notice a pattern (Figure 1.20). For example, there is only one way to reach each of the points on the two outside edges; there are two ways to reach the middle point in the row of points labeled 1, 2, 1; and so on. Observe that the point labeled 2 in Figure 1.20 can be found by adding the two 1s above it.

**Step 3** Carry Out the Plan

To see how many paths there are to any point, observe that you need only *add* the number of paths required to arrive at the point or points immediately above. To reach a point beneath the pair 1 and 2, the paths to 1 and 2 are extended downward, resulting in $1 + 2 = 3$ paths to that point. The resulting number pattern is shown in Figure 1.21. Notice, for example, that $4 + 6 = 10$ and $20 + 15 = 35$. (This pattern is part of what is called **Pascal’s triangle**. It is used again in Chapter 11.) The surrounded portion of this pattern applies to the given problem; thus the answer to the problem is 20.

**Step 4** Look Back

Can you see how to solve a similar problem involving a larger square array, say a $4 \times 4$ grid? How about a $10 \times 10$ grid? How about a rectangular grid?

A pattern of numbers arranged in a particular order is called a number **sequence**, and the individual numbers in the sequence are called **terms** of the sequence. The **counting numbers**, 1, 2, 3, 4, \ldots, give rise to many sequences. (An **ellipses**, the three periods after the 4, means “and so on.”) Several sequences of counting numbers follow.

| SEQUENCE \(2, 4, 6, \ldots\) | NAME \(\text{The even (counting) numbers}\) |
| SEQUENCE \(1, 3, 5, 7, \ldots\) | NAME \(\text{The odd (counting) numbers}\) |
| SEQUENCE \(1, 4, 9, 16, \ldots\) | NAME \(\text{The square (counting) numbers}\) |
| SEQUENCE \(1, 3, 9, 27, \ldots\) | NAME \(\text{The powers of three}\) |
| SEQUENCE \(1, 1, 2, 3, 5, 8, \ldots\) | NAME \(\text{The Fibonacci sequence (after the two 1s, each term is the sum of the two preceding terms)}\) |
Section 1.2  Three Additional Strategies

NCTM Standard
All students should analyze how both repeating and growing patterns are generated.

Reflection from Research
In classrooms where problem solving is valued and teachers have knowledge of children's mathematical thinking, children see mathematics as a problem-solving endeavor in which communicating mathematical thinking is important (Franke & Carey, 1997).

Inductive reasoning is used to draw conclusions or make predictions about a large collection of objects or numbers, based on a small representative subcollection. For example, inductive reasoning can be used to find the ones digit of the 400th term of the sequence 8, 12, 16, 20, 24, ... By continuing this sequence for a few more terms, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, ... one can observe that the ones digit of every fifth term starting with the term 24 is a four. Thus, the ones digit of the 400th term must be a four.

Additional Problems Where the Strategy “Look for a Pattern” Is Useful

1. Find the ones digit in $3^{99}$.

   **Step 1** Understand the Problem
   The number $3^{99}$ is the product of 99 threes. Using the exponent key on one type of scientific calculator yields the result $1.71792506547$. This shows the first digit, but not the ones (last) digit, since the 47 indicates that there are 47 places to the right of the decimal. (See the discussion on scientific notation in Chapter 4 for further explanation.) Therefore, we will need to use another method.

   **Step 2** Devise a Plan
   Consider $3^1, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, 3^8$. Perhaps the ones digits of these numbers form a pattern that can be used to predict the ones digit of $3^{99}$.

   **Step 3** Carry Out the Plan
   $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243, 3^6 = 729, 3^7 = 2187, 3^8 = 6561$. The ones digits form the sequence 3, 9, 7, 1, 3, 9, 7, 1. Whenever the exponent of the 3 has a factor of 4, the ones digit is a 1. Since 100 has a factor of 4, $3^{100}$ must have a ones digit of 1. Therefore, the ones digit of $3^{99}$ must be 7, since $3^{99}$ precedes $3^{100}$ and 7 precedes 1 in the sequence 3, 9, 7, 1.

   **Step 4** Look Back
   Ones digits of other numbers involving exponents might be found in a similar fashion. Check this for several of the numbers from 4 to 9.

2. Which whole numbers, from 1 to 50, have an odd number of factors? For example, 15 has 1, 3, 5, and 15 as factors, and hence has an even number of factors: four.

3. In the next diagram, the left “H”-shaped array is called the 32-H and the right array is the 58-H.
Chapter 1  Introduction to Problem Solving

**Reflection from Research**
Problem-solving ability develops with age, but the relative difficulty inherent in each problem is grade independent (Christou & Philippou, 1998).

**Children's Literature**
www.wiley.com/college/musser
See "Patterns in Peru" by Cindy Neuschwander.

**NCTM Standard**
Instructional programs should enable all students to build new mathematical knowledge through problem solving.

---

**Problem**
The number 10 can be expressed as the sum of four odd numbers in three ways:
(i) \(10 = 7 + 1 + 1 + 1\), (ii) \(10 = 5 + 3 + 1 + 1\), and (iii) \(10 = 3 + 3 + 3 + 1\). In how many ways can 20 be expressed as the sum of eight odd numbers?

---

**Clues**
The Look for a Pattern strategy may be appropriate when
- A list of data is given.
- A sequence of numbers is involved.
- Listing special cases helps you deal with complex problems.
- You are asked to make a prediction or generalization.
- Information can be expressed and viewed in an organized manner, such as in a table.

Review the preceding three problems to see how these clues may have helped you select the Look for a Pattern strategy to solve these problems.

---

**Strategy 5** Make a List
The Make a List strategy is often combined with the Look for a Pattern strategy to suggest a solution to a problem. For example, here is a list of all the squares of the numbers 1 to 20 with their ones digits in boldface.

\[
\begin{align*}
1 & \quad 1 \\
4 & \quad 16 \\
9 & \quad 81 \\
16 & \quad 256 \\
25 & \quad 625 \\
36 & \quad 8100 \\
49 & \quad 2401 \\
64 & \quad 1296 \\
81 & \quad 9801 \\
100 & \quad 10000 \\
121 & \quad 144 \\
144 & \quad 169 \\
169 & \quad 225 \\
196 & \quad 256 \\
225 & \quad 289 \\
256 & \quad 324 \\
289 & \quad 361 \\
324 & \quad 400
\end{align*}
\]

The pattern in this list can be used to see that the ones digits of squares must be one of 0, 1, 4, 5, 6, or 9. This list suggests that a perfect square can never end in a 2, 3, 7, or 8.

---

**a.** Find the sums of the numbers in the 32-H. Do the same for the 58-H and the 74-H. What do you observe?

**b.** Find an H whose sum is 497.

**c.** Can you predict the sum in any H if you know the middle number? Explain.
Reflection from Research
Correct answers are not a safe indicator of good thinking. Teachers must examine more than answers and must demand from students more than answers (Sowder, Threadgill-Sowder, Moyer, & Moyer, 1983).

Additional Problems Where the Strategy “Make a List” Is Useful

1. In a dart game, three darts are thrown. All hit the target (Figure 1.22). What scores are possible?

   Step 1 Understand the Problem
   Assume that all three darts hit the board. Since there are four different numbers on the board, namely, 0, 1, 4, and 16, three of these numbers, with repetitions allowed, must be hit.

   Step 2 Devise a Plan
   We should make a systematic list by beginning with the smallest (or largest) possible sum. In this way we will be more likely to find all sums.

   Step 3 Carry Out the Plan
   \[ 0 + 0 + 0 = 0, \quad 0 + 0 + 1 = 1, \quad 0 + 1 + 1 = 2, \]
   \[ 1 + 1 + 1 = 3, \quad 0 + 0 + 4 = 4, \quad 0 + 1 + 4 = 5, \]
   \[ 1 + 1 + 4 = 6, \quad 0 + 4 + 4 = 8, \quad 1 + 4 + 4 = 9, \]
   \[ 4 + 4 + 4 = 12, \quad \ldots, \quad 16 + 16 + 16 = 48. \]

   Step 4 Look Back
   Could you have used the three sums to 10 to help find these 11 sums to 20? Can you think of similar problems to solve? For example, an easier one would be to express 8 as the sum of four odd numbers, and a more difficult one would be to express 40 as the sum of 16 odd numbers. We could also consider sums of even numbers, expressing 20 as the sum of six even numbers.

2. How many squares, of all sizes, are there on an 8 \times 8 checkerboard? (See Figure 1.23; the sides of the squares are on the lines.)

3. It takes 1230 numerical characters to number the pages of a book. How many pages does the book contain?
Step 1

Understand the Problem

Coins may be placed on both pans. If one side of the balance is lower than the other, that side contains the heavier coin. If a coin is placed in each pan and the pans balance, the heavier coin is in the remaining seven. We could continue in this way, but if we missed the heavier coin each time we tried two more coins, the last coin would be the heavy one. This would require four balancings. Can we find the heavier coin in fewer balancings?

Step 2

Devise a Plan

To find a more efficient method, let’s examine the cases of three coins and five coins before moving to the case of nine coins.

Step 3

Carry Out the Plan

Three coins: Put one coin on each pan (Figure 1.24). If the pans balance, the third coin is the heavier one. If they don’t, the one in the lower pan is the heavier one. Thus, it only takes one balancing to find the heavier coin.

Five coins: Put two coins on each pan (Figure 1.25). If the pans balance, the fifth coin is the heavier one. If they don’t, the heavier one is in the lower pan. Remove the two coins in the higher pan and put one of the two coins in the lower pan on the other pan. In this case, the lower pan will have the heavier coin. Thus, it takes at most two balancings to find the heavier coin.
Nine coins: At this point, patterns should have been identified that will make this solution easier. In the three-coin problem, it was seen that a heavy coin can be found in a group of three as easily as it can in a group of two. From the five-coin problem, we know that by balancing groups of coins together, we could quickly reduce the number of coins that needed to be examined. These ideas are combined in the nine-coin problem by breaking the nine coins into three groups of three and balancing two groups against each other (Figure 1.26). In this first balancing, the group with the heavy coin is identified. Once the heavy coin has been narrowed to three choices, then the three-coin balancing described above can be used.

The minimum number of balancings needed to locate the heavy coin out of a set of nine coins is two.

**Step 4** Look Back
In solving this problem by using simpler problems, no numerical patterns emerged. However, patterns in the balancing process that could be repeated with a larger number of coins did emerge.

### Additional Problems Where the Strategy “Solve a Simpler Problem” Is Useful

1. Find the sum \( \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{10}} \).

**Step 1** Understand the Problem
This problem can be solved directly by getting a common denominator, here \(2^{10}\), and finding the sum of the numerators.

**Step 2** Devise a Plan
Instead of doing a direct calculation, let's combine some previous strategies. Namely, make a list of the first few sums and look for a pattern.

**Step 3** Carry Out the Plan
\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{3}{4}, \quad \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16}, \quad \frac{1}{8} + \frac{1}{16} = \frac{15}{16}
\]

The pattern of sums, \( \frac{1}{4}, \frac{3}{8}, \frac{7}{16} \), suggests that the sum of the 10 fractions is \( \frac{2^{10} - 1}{2^{10}} \), or \( \frac{1023}{1024} \).

**Step 4** Look Back
This method of combining the strategy of Solve a Simpler Problem with Make a List and Look for a Pattern is very useful. For example, what is the sum \( \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{100}}? \) Because of the large denominators, you wouldn’t want to add these fractions directly.
2. Following the arrows in Figure 1.27, how many paths are there from A to B?

Figure 1.27

3. There are 20 people at a party. If each person shakes hands with each other person, how many handshakes will there be?

CLUES

The Solve a Simpler Problem strategy may be appropriate when

- The problem involves complicated computations.
- The problem involves very large or very small numbers.
- A direct solution is too complex.
- You want to gain a better understanding of the problem.
- The problem involves a large array or diagram.

Review the preceding three problems to see how these clues may have helped you select the Solve a Simpler Problem strategy to solve these problems.

Combining Strategies to Solve Problems

As shown in the previous four-step solution, it is often useful to employ several strategies to solve a problem. For example, in Section 1.1, a pizza problem similar to the following was posed: What is the maximum number of pieces you can cut a pizza into using four straight cuts? This question can be extended to the more general question: What is the maximum number of pieces you can cut a pizza into using \( n \) straight cuts? To answer this, consider the sequence in Figure 1.9: 1, 2, 4, 7, 11. To identify patterns in a sequence, observing how successive terms are related can be helpful. In this case, the second term of 2 can be obtained from the first term of 1 by either adding 1 or multiplying by 2. The third term, 4, can be obtained from the second term, 2, by adding 2 or multiplying by 2. Although multiplying by 2 appears to be a pattern, it fails as we move from the third term to the fourth term. The fourth term can be found by adding 3 to the third term. Thus, the sequence appears to be the following:

<table>
<thead>
<tr>
<th>Sequence</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>7</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Differences</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Extending the difference sequence, we obtain the following

<table>
<thead>
<tr>
<th>Term</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>...</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sequence</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>11</td>
<td>16</td>
<td>22</td>
<td>29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Differences</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Algebraic Reasoning
In order to generalize patterns, one must first identify what is staying the same (the initial 1) and what is changing (the difference is increasing by one) with each step. Secondly, one must notice the relationship between what is changing and the step number.

Starting with 1 in the sequence, dropping down to the difference line, then back up to the number in the sequence line, we find the following:

1st Term: $1 = 1$
2nd Term: $2 = 1 + 1$
3rd Term: $4 = 1 + (1 + 2)$
4th Term: $7 = 1 + (1 + 2 + 3)$
5th Term: $11 = 1 + (1 + 2 + 3 + 4)$, and so forth

Recall that earlier we saw that $1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}$. Thus, the $n$th term in the sequence is $1 + [1 + 2 + \ldots + (n - 1)] = 1 + \frac{(n - 1)n}{2}$. Notice that as a check, the eighth term in the sequence is $1 + \frac{7 \cdot 8}{2} = 1 + 28 = 29$. Hence, to solve the original problem, we used Draw a Picture, Look for a Pattern, and Use a Variable.

It may be that a pattern does not become obvious after one set of differences. Consider the following problem where several differences are required to expose the pattern.

**Problem** If 10 points are placed on a circle and each pair of points is connected with a segment, what is the maximum number of regions created by these segments?

**Step 1** Understand the Problem
This problem can be better understood by drawing a picture. Since drawing 10 points and all of the joining segments may be overwhelming, looking at a simpler problem of circles with 1, 2, or 3 points on them may help in further understanding the problem. The first three cases are in Figure 1.28.

![Figure 1.28](image)

**Step 2** Devise a Plan
So far, the number of regions are 1, 2, and 4 respectively. Here, again, this could be the start of the pattern, 1, 2, 4, 8, 16, \ldots Let’s draw three more pictures to see if this is the case. Then the pattern can be generalized to the case of 10 points.

**Step 3** Carry Out the Plan
The next three cases are shown in Figure 1.29.
Making a list of the number of points on the circle and the corresponding number of regions will help us see the pattern.

<table>
<thead>
<tr>
<th>Points</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regions</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>31</td>
</tr>
</tbody>
</table>

While the pattern for the first five cases makes it appear as if the number of regions are just doubling with each additional point, the 31 regions with 6 points ruins this pattern. Consider the differences between the numbers in the pattern and look for a pattern in the differences.

<table>
<thead>
<tr>
<th>Points</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regions</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>31</td>
</tr>
</tbody>
</table>

1st Difference | 1 | 2 | 4 | 8 | 15
2nd Difference | 1 | 2 | 4 | 7
3rd Difference | 1 | 2 | 3
4th Difference | 1 | 1

Because the first, second, and third difference did not indicate a clear pattern, the fourth difference was computed and revealed a pattern of all ones. This observation is used to extend the pattern by adding four 1s to the two 1s in the fourth difference to make a sequence of six 1s. Then we work up until the Regions sequence has ten numbers as shown next.

<table>
<thead>
<tr>
<th>Points</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regions</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>31</td>
<td>57</td>
<td>99</td>
<td>163</td>
<td>256</td>
</tr>
</tbody>
</table>

1st Difference | 1 | 2 | 4 | 8 | 15 | 26 | 42 | 64 | 95 |
2nd Difference | 1 | 2 | 4 | 7 | 11 | 16 | 22 | 29 |
3rd Difference | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
4th Difference | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
By finding successive differences, it can be seen that the solution to our problem is 256 regions.

**Step 4** Look Back

By using a combination of Draw a Picture, Solve a Simpler Problem, Make a List, and Look for a Pattern, the solution was found. It can also be seen that it is important when looking for such patterns to realize that we may have to look at many terms and many differences to be able to find the pattern.

**Recapitulation**

When presenting the problems in this chapter, we took great care in organizing the solutions using Pólya’s four-step approach. However, it is not necessary to label and display each of the four steps every time you work a problem. On the other hand, it is good to get into the habit of recalling the four steps as you plan and as you work through a problem. In this chapter we have introduced several useful problem-solving strategies. In each of the following chapters, a new problem-solving strategy is introduced. These strategies will be especially helpful when you are making a plan. As you are planning to solve a problem, think of the strategies as a collection of tools. Then an important part of solving a problem can be viewed as selecting an appropriate tool or strategy.

We end this chapter with a list of suggestions that students who have successfully completed a course on problem solving felt were helpful tips. Reread this list periodically as you progress through the book.

**Suggestions from Successful Problem Solvers**

- Accept the challenge of solving a problem.
- Rewrite the problem in your own words.
- Take time to explore, reflect, think....
- Talk to yourself. Ask yourself lots of questions.
- If appropriate, try the problem using simple numbers.
- Many problems require an incubation period. If you get frustrated, do not hesitate to take a break—your subconscious may take over. But do return to try again.
- Look at the problem in a variety of ways.
- Run through your list of strategies to see whether one (or more) can help you get a start.
- Many problems can be solved in a variety of ways—you only need to find one solution to be successful.
- Do not be afraid to change your approach, strategy, and so on.
- Organization can be helpful in problem solving. Use the Pólya four-step approach with a variety of strategies.
- Experience in problem solving is very valuable. Work lots of problems; your confidence will grow.
- If you are not making much progress, do not hesitate to go back to make sure that you really understand the problem. This review process may happen two or three times in a problem since understanding usually grows as you work toward a solution.
- There is nothing like a breakthrough, a small *aha!*, as you solve a problem.
• Always, always look back. Try to see precisely what the key step was in your solution.
• Make up and solve problems of your own.
• Write up your solutions neatly and clearly enough so that you will be able to un-
derstand your solution if you reread it in 10 years.
• Develop good problem-solving helper skills when assisting others in solving problems. Do not give out solutions; instead, provide meaningful hints.
• By helping and giving hints to others, you will find that you will develop many new insights.
• Enjoy yourself! Solving a problem is a positive experience.

Sophie Germain was born in Paris in 1776, the daughter of a silk merchant. At the age of 13, she found a book on the history of mathematics in her father's library. She became enthralled with the study of mathematics. Even though her parents disapproved of this pursuit, nothing daunted her—she studied at night wrapped in a blanket, because her parents had taken her clothing away from her to keep her from getting up. They also took away her heat and light. This only hardened her resolve until her father finally gave in and she, at last, was allowed to study to become a mathematician.

### Section 1.2 EXERCISE / PROBLEM SET A

Use any of the six problem-solving strategies introduced thus far to solve the following.

1. a. Complete this table and describe the pattern in the 'Answer' column.

<table>
<thead>
<tr>
<th>SUM</th>
<th>ANSWER</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1 + 3</td>
<td>4</td>
</tr>
<tr>
<td>1 + 3 + 5</td>
<td>10</td>
</tr>
<tr>
<td>1 + 3 + 5 + 7</td>
<td>21</td>
</tr>
<tr>
<td>1 + 3 + 5 + 7 + 9</td>
<td>36</td>
</tr>
</tbody>
</table>

b. How many odd whole numbers would have to be added to get a sum of 81? Check your guess by adding them.

c. How many odd whole numbers would have to be added to get a sum of 169? Check your guess by adding them.

d. How many odd whole numbers would have to be added to get a sum of 529? (You do not need to check.)

2. Find the missing term in each pattern.

   a. 256, 128, 64, ____ , 16, 8
   b. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$
   c. 7, 9, 12, 16, ____
   d. 127,863; 12,789; ____ ; 135; 18

3. Sketch a figure that is next in each sequence.

   a. [Grid with A and B]...[Grid with C]
   b. [Grid with arrows and circles]
   c. [Grid with arrows and circles]
Section 1.2  Three Additional Strategies

4. Consider the following differences. Use your calculator to verify that the statements are true.

\[ 6^2 - 5^2 = 11 \]
\[ 56^2 - 45^2 = 1111 \]
\[ 556^2 - 445^2 = 111,111 \]

a. Predict the next line in the sequence of differences. Use your calculator to check your answer.

b. What do you think the eighth line will be?

5. Look for a pattern in the first two number grids. Then use the pattern you observed to fill in the missing numbers of the third grid.

\[
\begin{array}{cccc}
21 & 7 & 3 & 72 \ \\
3 & 1 & 3 & 8 \ \\
7 & 7 & 1 & 9 \ \\
\end{array}
\]
\[
\begin{array}{cccc}
60 & 6 & 2 & 5 \ \\
8 & 4 & 2 & \ \\
9 & 9 & 1 & \ \\
\end{array}
\]

6. The triangular numbers are the whole numbers that are represented by certain triangular arrays of dots. The first five triangular numbers are shown.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \ \\
\end{array}
\]

a. Complete the following table and describe the pattern in the ‘Number of Dots’ column.

<table>
<thead>
<tr>
<th>NUMBER</th>
<th>NUMBER OF DOTS (TRIANGULAR NUMBERS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

b. Make a sketch to represent the seventh triangular number.

c. How many dots will be in the tenth triangular number?

d. Is there a triangular number that has 91 dots in its shape? If so, which one?

e. Is there a triangular number that has 150 dots in its shape? If so, which one?

f. Write a formula for the number of dots in the \( n \)th triangular number.

g. When the famous mathematician Carl Friedrich Gauss was in fourth grade, his teacher challenged him to add the first one hundred counting numbers. Find this sum.

\[ 1 + 2 + 3 + \ldots + 100 \]

7. In a group of 12 coins identical in appearance, all weigh the same except one that is heavier. What is the minimum number of weighings required to determine the counterfeit coin? Use the Chapter 1 eManipulative activity Counterfeit Coin on our Web site for eight or nine coins to better understand the problem.

8. If 20 points are placed on a circle and every pair of points is joined with a segment, what is the total number of segments drawn?

9. Find reasonable sixth, seventh, and eighth terms of the following sequences:
   a. 1, 4, 9, 17, 29, _____, _____, _____
   b. 3, 7, 13, 21, 31, _____, _____

10. As mentioned in this section, the square numbers are the counting numbers 1, 4, 9, 16, 25, 36, .... Each square number can be represented by a square array of dots as shown in the following figure, where the second square number has four dots, and so on. The first four square numbers are shown.

11. Would you rather work for a month (30 days) and get paid 1 million dollars or be paid 1 cent the first day, 2 cents the second day, 4 cents the third day, and so on? Explain.

12. Find the perimeters and then complete the table.

\[
\begin{array}{cccccc}
\text{number of triangles} & 1 & 2 & 3 & 4 & 5 & 6 & 10 & n \\
\text{perimeter} & \text{perimeter} & \text{perimeter} & \text{perimeter} & 40 \ \\
\end{array}
\]
13. The integers greater than 1 are arranged as shown.

```
  2  3  4  5
 9  8  7  6
10 11 12 13
17 16 15 14
  .  .  .  .
```

a. In which column will 100 fall?
b. In which column will 1000 fall?
c. How about 1999?
d. How about 99,997?

14. How many cubes are in the 100th collection of cubes in this sequence?

```

```

15. The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, . . . ,
where each successive number beginning with 2 is the sum of the preceding two; for example, $13 = 5 + 8, 21 = 8 + 13,$ and so on. Observe the following pattern.

\[
\begin{align*}
1^2 + 1^2 &= 1 \times 2 \\
1^2 + 1^2 + 2^2 &= 2 \times 3 \\
1^2 + 1^2 + 2^2 + 3^2 &= 3 \times 5
\end{align*}
\]

Write out six more terms of the Fibonacci sequence and use the sequence to predict what \(12/11001\) \(22/11001\) \(32/11001\) \(\ldots\) \(1442\) is without actually computing the sum. Then use your calculator to check your result.

16. Write out 16 terms of the Fibonacci sequence and observe the following pattern:

\[
\begin{align*}
1 + 2 &= 3 \\
1 + 2 + 5 &= 8 \\
1 + 2 + 5 + 13 &= 21
\end{align*}
\]

Use the pattern you observed to predict the sum

\[
1 + 2 + 5 + 13 + \ldots + 610
\]

without actually computing the sum. Then use your calculator to check your result.

17. Pascal’s triangle is where each entry other than a 1 is obtained by adding the two entries in the row above it.

```
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
  .  .  .  .  .
```

a. Find the sums of the numbers on the diagonals in Pascal’s triangle as are indicated in the following figure.

b. Predict the sums along the next three diagonals in Pascal’s triangle without actually adding the entries. Check your answers by adding entries on your calculator.

18. Answer the following questions about Pascal’s triangle (see Problem 17).

a. In the triangle shown here, one number, namely 3, and the six numbers immediately surrounding it are encircled. Find the sum of the encircled seven numbers.

```
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
  .  .  .  .  .  .  .
```

b. Extend Pascal’s triangle by adding a few rows. Then draw several more circles anywhere in the triangle like the one shown in part (a). Explain how the sums obtained by adding the seven numbers inside the circle are related to one of the numbers outside the circle.

19. Consider the following sequence of shapes. The sequence starts with one square. Then at each step squares are attached around the outside of the figure, one square per exposed edge in the figure.

```
Step 1

Step 2

Step 3

Step 4

```

a. Find the sum of the numbers on the diagonals in Pascal’s triangle as are indicated in the following figure.
Section 1.2 Three Additional Strategies

a. Draw the next two figures in the sequence.
b. Make a table listing the number of unit squares in the figure at each step. Look for a pattern in the number of unit squares. *(Hint: Consider the number of squares attached at each step.)*
c. Based on the pattern you observed, predict the number of squares in the figure at step 7. Draw the figure to check your answer.
d. How many squares would there be in the 10th figure? in the 20th figure?

20. In a dart game, only 4 points or 9 points can be scored on each dart. What is the largest score that it is not possible to obtain? (Assume that you have an unlimited number of darts.)

21. If the following four figures are referred to as stars, the first one is a three-pointed star and the second one is a six-pointed star. *(Note: If this pattern of constructing a new equilateral triangle on each side of the existing equilateral triangle is continued indefinitely, the resulting figure is called the Koch curve or Koch snowflake.)*

a. How many points are there in the third star?
b. How many points are there in the fourth star?

22. Using the Chapter 1 eManipulative activity Color Patterns on our Web site, describe the color patterns for the first three computer exercises.

23. Looking for a pattern can be frustrating if the pattern is not immediately obvious. Create your own sequence of numbers that follows a pattern but that has the capacity to stump some of your fellow students. Then write an explanation of how they might have been able to discover your pattern.

**Section 1.2 EXERCISE / PROBLEM SET B**

1. Find the missing term in each pattern.
   a. 10, 17, ____ , 37, 50, 65
   b. 1, 2 , 7, 9
   c. 243, 324, 405, ____ , 567
   d. 234, ____ , 23,481; 234,819; 2,348,200

2. Sketch a figure that is next in each sequence.
   a. 
   b. 

3. The rectangular numbers are whole numbers that are represented by certain rectangular arrays of dots. The first five rectangular numbers are shown.

   a. Complete the following table and describe the pattern in the ‘Number of Dots’ column.
   b. Make a sketch to represent the seventh rectangular number.
   c. How many dots will be in the tenth rectangular number?
   d. Is there a rectangular number that has 380 dots in its shape? If so, which one?
   e. Write a formula for the number of dots in the \( n \)th rectangular number.
   f. What is the connection between triangular numbers (see Problem 6 in Part A) and rectangular numbers?
4. The **pentagonal numbers** are whole numbers that are represented by pentagonal shapes. The first four pentagonal numbers are shown.

   ![Pentagonal Numbers Diagram]

   a. Complete the following table and describe the pattern in the ‘Number of Dots’ column.

<table>
<thead>
<tr>
<th>NUMBER</th>
<th>NUMBER OF DOTS (PENTAGONAL NUMBERS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

b. Make a sketch to represent the fifth pentagonal number.

c. How many dots will be in the ninth pentagonal number?

d. Is there a pentagonal number that has 200 dots in its shape? If so, which one?

e. Write a formula for the number of dots in the \( n \)th pentagonal number.

5. Consider the following process.

   i. Choose a whole number.
   
   ii. Add the squares of the digits of the number to get a new number.

   Repeat step 2 several times.

   a. Apply the procedure described to the numbers 12, 13, 19, 21, and 127.
   
   b. What pattern do you observe as you repeat the steps over and over?
   
   c. Check your answer for part (b) with a number of your choice.

6. How many triangles are in the picture?

7. What is the smallest number that can be expressed as the sum of two squares in two different ways? (You may use one square twice.)

8. How many cubes are in the 10th collection of cubes in this sequence?

9. The \( 2 \times 2 \) array of numbers \[
\begin{bmatrix}
  4 & 5 \\
  5 & 6
\end{bmatrix}
\]
has a sum of \( 4 \times 5 \),
and the \( 3 \times 3 \) array
\[
\begin{bmatrix}
  6 & 7 & 8 \\
  7 & 8 & 9 \\
  8 & 9 & 10
\end{bmatrix}
\]
has a sum of \( 9 \times 8 \).

   a. What will be the sum of the similar \( 4 \times 4 \) array starting with ??
   
   b. What will be the sum of a similar \( 10 \times 10 \) array starting with 100?

10. The Fibonacci sequence was defined to be the sequence 1, 1, 2, 3, 5, 8, 13, 21, ... where each successive number is the sum of the preceding two. Observe the following pattern.

   \[
   \begin{align*}
   1 + 1 & = 3 - 1 \\
   1 + 1 + 2 & = 5 - 1 \\
   1 + 1 + 2 + 3 & = 8 - 1 \\
   1 + 1 + 2 + 3 + 5 & = 13 - 1
   \end{align*}
   \]

   Write out six more terms of the Fibonacci sequence, and use the sequence to predict the answer to

   \[
   1 + 1 + 2 + 3 + 5 + \ldots + 144
   \]

   without actually computing the sum. Then use your calculator to check your result.

11. Write 16 terms of the Fibonacci sequence.

   a. Notice that the fourth term in the sequence (called \( F_4 \)) is odd: \( F_4 = 3 \). The sixth term in the sequence (called \( F_6 \)) is even: \( F_6 = 8 \). Look for a pattern in the terms of the sequence, and describe which terms are even and which are odd.

   b. Which of the following terms of the Fibonacci sequence are even and which are odd: \( F_{30}, F_{51}, F_{199}, F_{200}, F_{300} \)?

   c. Look for a pattern in the terms of the sequence and describe which terms are divisible by 3.

   d. Which of the following terms of the Fibonacci sequence are multiples of 3: \( F_{48}, F_{75}, F_{196}, F_{379}, F_{1000} \)?

12. Write out 16 terms of the Fibonacci sequence and observe the following pattern.

   \[
   \begin{align*}
   1 + 3 & = 5 - 1 \\
   1 + 3 + 8 & = 13 - 1 \\
   1 + 3 + 8 + 21 & = 34 - 1
   \end{align*}
   \]

   Use the pattern you observed to predict the answer to

   \[
   1 + 3 + 8 + 21 + \ldots + 377
   \]

   without actually computing the sum. Then use your calculator to check your result.
13. Investigate the “Tower of Hanoi” problem on the Chapter 1 eManipulative activity Tower of Hanoi on our Web site to answer the following questions:
   a. Determine the fewest number of moves required when you start with two, three, and four disks.
   b. Describe the general process to move the disks in the fewest number of moves.
   c. What is the minimum number of moves that it should take to move six disks?

14. While only 19 years old, Carl Friedrich Gauss proved in 1796 that every positive integer is the sum of at the most three triangular numbers (see Problem 6 in Part A).
   a. Express each of the numbers 25 to 35 as a sum of no more than three triangular numbers.
   b. Express the numbers 74, 81, and 90 as sums of no more than three triangular numbers.

15. Answer the following for Pascal’s triangle.
   a. In the following triangle, six numbers surrounding a central number, 4, are circled. Compare the products of alternate numbers moving around the circle; that is, compare $3 \cdot 1 \cdot 10$ and $6 \cdot 1 \cdot 5$.

\[
\begin{array}{cccc}
1 \\
1 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}
\]

   b. Extend Pascal’s triangle by adding a few rows. Then draw several more circles like the one shown in part (a) anywhere in the triangle. Find the products as described in part (a). What patterns do you see in the products?

16. A certain type of gutter comes in 6-foot, 8-foot, and 10-foot sections. How many different lengths can be formed using three sections of gutter?

17. Consider the sequence of shapes shown in the following figure. The sequence starts with one triangle. Then at each step, triangles are attached to the outside of the preceding figure, one triangle per exposed edge.

   a. Draw the next two figures in the sequence.
   b. Make a table listing the number of triangles in the figure at each step. Look for a pattern in the number of triangles. (Hint: Consider the number of triangles added at each step.)
   c. Based on the pattern you observed, predict the number of triangles in the figure at step 7. Draw the figure to check your answer.
   d. How many triangles would there be in the 10th figure? in the 20th figure? in the 50th figure?

18. How many equilateral triangles of all sizes are there in the $3 \times 3 \times 3$ equilateral triangle shown next?

19. Refer to the following figures to answer the questions.
   (Note: If this pattern is continued indefinitely, the resulting figure is called the Sierpinski triangle or the Sierpinski gasket.)

   a. How many black triangles are there in the fourth figure?
   b. How many white triangles are there in the fourth figure?
   c. If the pattern is continued, how many black triangles are there in the $n$th figure?
   d. If the pattern is continued, how many white triangles are there in the $n$th figure?

20. If the pattern illustrated next is continued.
   a. find the total number of 1 by 1 squares in the thirtieth figure.
   b. find the perimeter of the twenty-fifth figure.
   c. find the total number of toothpicks used to construct the twentieth figure.
21. Find reasonable sixth, seventh, and eighth terms of the following sequences:
   a. 1, 3, 4, 7, 11, _____, _____, _____
   b. 0, 1, 4, 12, 29, _____, _____, _____

22. Many board games involve throwing two dice and summing of the numbers that come up to determine how many squares to move. Make a list of all the different sums that can appear. Then write down how many ways each different sum can be formed. For example, 11 can be formed in two ways: from a 5 on the first die and a 6 on the second OR a 6 on the first die and a 5 on the second. Which sum has the greatest number of combinations? What conclusion could you draw from that?

23. There is an old riddle about a frog at the bottom of a 20-foot well. If he climbs up 3 feet each day and slips back 2 feet each night, how many days will it take him to reach the 20-foot mark and climb out of the well? The answer isn’t 20. Try doing the problem with a well that is only 5 feet deep, and keep track of all the frog’s moves. What strategy are you using?

Analyzing Student Thinking

24. Marietta extended the pattern 2, 4, 8 to be 2, 4, 8, 16, 32, . . . . Pascuel extended the same pattern to be 2, 4, 8, 14, 22, . . . . They asked you who was correct. How should you respond?

25. Eula is asked to find the following sum: $1 + 3 + 5 + 7 + \cdots + 97 + 99$. She decided to “solve a simpler problem” but doesn’t know where to start. What would you suggest?

26. When Mickey counted the number of outcomes of rolling two 4-sided, tetrahedral dice (see Figure 12.89 for an example of a tetrahedron), he decided to “make a list” and got $(1,1), (2,2), (3,3), (4,4), (1,2), (3,4), (2,4), (4,1)$. He started to get confused about which ones he had listed and which ones were left. How would you help him create a more systematic list?

27. Bridgette says that she knows how to see patterns like 3, 7, 11, 15, . . . and 5, 12, 19, 26, 33, . . . because “you are just adding the same amount each time.” However, she can’t see the pattern in 3, 5, 9, 15, 23 because the difference between the numbers is changing. How could you help her see this new pattern?

28. Jeremy is asked to find the number of rectangles of all possible dimensions in the figure below and decides to “solve a simpler problem.”

![Diagram of a grid of rectangles]

What simpler problem would you suggest he use?

29. After solving several simpler problems, Janell commented that she often makes a list of solutions of the simpler problems and then looks for a pattern in that list. She asks if it is okay to use more than one type of strategy when solving the same problem. How should you respond?

Problems Related to the NCTM Standards and Curriculum Focal Points

1. The Focal Points for Grade 3 state “Apply increasingly sophisticated strategies . . . to solve multiplication and division problems.” Find a problem in this problem set that illustrates this statement and explain your reasoning.

2. The NCTM Standards state “Instructional programs should enable all students to build new mathematical knowledge through problem solving.” Explain some new mathematical knowledge that you have learned by solving problems in this section.

3. The NCTM Standards state “All students should represent, analyze, and generalize a variety of patterns with tables, graphs, and, when possible, symbolic rules.” Explain how tables, graphs, and symbolic rules could be used to represent and analyze the pattern in Problem 11 of Problem Set A.
**END OF CHAPTER MATERIAL**

**Solution of Initial Problem**
Place the whole numbers 1 through 9 in the circles in the accompanying triangle so that the sum of the numbers on each side is 17.

```
  1
 6 4
8 9
2 5 7 3 2 4 8 3
```

**Strategy: Guess and Test**
Having solved a simpler problem in this chapter, you might easily be able to conclude that 1, 2, and 3 must be in the corners. Then the remaining six numbers, 4, 5, 6, 7, 8, and 9, must produce three pairs of numbers whose sums are 12, 13, and 14. The only two possible solutions are as shown.

```
1
6 4
8 9
2 5 7 3 2 4 8 3
```

**Solution for Additional Problems**

**Guess and Test**
1. \( s = 1, a = 3, n = 6, f = 9, w = 0, i = 7, m = 2 \)
2. \( 0 = (4 - 4) + (4 - 4) \)
   \( 1 = (4 + 4) ÷ (4 + 4) \)
   \( 2 = (4 + 4) + (4 + 4) \)
   \( 3 = (4 + 4 + 4) ÷ 4 \)
   \( 4 = 4 + 4 × (4 - 4) \)
   \( 5 = (4 × 4 + 4) ÷ 4 \)
   \( 6 = ((4 + 4) ÷ 4) + 4 \)
   \( 7 = 4 + 4 - (4 + 4) \)
   \( 8 = ((4 × 4) ÷ 4) + 4 \)
   \( 9 = 4 + 4 + (4 + 4) \)
3. There are many other possible answers.

**Draw a Picture**
1. 5
2. Yes; make one cut, then lay the logs side by side for the second cut.
3. 12

**Use a Variable**
1. 55, 5050, 125, 250
2. \((2m + 1) + (2m + 3) + (2m + 5) + (2m + 7) + (2m + 9) = 10m + 25 = 5(2m + 5)\)
3. 10°, 80°, 90°

**Look for a Pattern**
1. 7
2. Square numbers
3. a. 224; 406; 518 b. 71 c. The sum is seven times the middle number.

**Make a List**
1. 48, 36, 33, 32, 24, 21, 20, 18, 17, 16, 12, 9, 8, 6, 5, 4, 3, 2, 1, 0
2. 204
3. 446

**Solve a Simpler Problem**
1. 0
2. 377
3. 190
People in Mathematics

Carl Friedrich Gauss (1777–1855)
Carl Friedrich Gauss, according to the historian E. T. Bell, “lives everywhere in mathematics.” His contributions to geometry, number theory, and analysis were deep and wide-ranging. Yet he also made crucial contributions in applied mathematics. When the tiny planet Ceres was discovered in 1800, Gauss developed a technique for calculating its orbit, based on meager observations of its direction from Earth at several known times. Gauss contributed to the modern theory of electricity and magnetism, and with the physicist W. E. Weber constructed one of the first practical electric telegraphs. In 1807 he became director of the astronomical observatory at Göttingen, where he served until his death. At age 18, Gauss devised a method for constructing a 17-sided regular polygon, using only a compass and straightedge. Remarkably, he then derived a general rule that predicted which regular polygons are likewise constructible.

Sophie Germain (1776–1831)
Sophie Germain, as a teenager in Paris, discovered mathematics by reading books from her father’s library. At age 18, Germain wished to attend the prestigious École Polytechnique in Paris, but women were not admitted. So she studied from classroom notes supplied by sympathetic male colleagues, and she began submitting written work using the pen name Antoine LeBlanc. This work won her high praise, and eventually she was able to reveal her true identity. Germain is noted for her theory of the vibration patterns of elastic plates and for her proof of Fermat’s last theorem in some special cases. Of Sophie Germain, Carl Gauss wrote, “When a woman, because of her sex, encounters infinitely more obstacles than men... yet overcomes these fetters and penetrates that which is most hidden, she doubtless has the most noble courage, extraordinary talent, and superior genius.”

CHAPTER REVIEW

Review the following terms and problems to determine which require learning or relearning—page numbers are provided for easy reference.

SECTION 1.1 The Problem-Solving Process and Strategies

VOCABULARY/NOTATION

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PROBLEMS

For each of the following, (i) determine a reasonable strategy to use to solve the problem, (ii) state a clue that suggested the strategy, and (iii) write out a solution using Pólya’s four-step process.

1. Fill in the circles using the numbers 1 through 9 once each where the sum along each of the five rows totals 17.
2. In the following arithmagon, the number that appears in a square is the product of the numbers in the circles on each side of it. Determine what numbers belong in the circles.

![Arithmagon Diagram]

3. The floor of a square room is covered with square tiles. Walking diagonally across the room from corner to corner, Susan counted a total of 33 tiles on the two diagonals. What is the total number of tiles covering the floor of the room?

**SECTION 1.2 Three Additional Strategies**

**VOCABULARY/NOTATION**

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**PROBLEMS**

For each of the following, (i) determine a reasonable strategy to use to solve the problem, (ii) state a clue that suggested the strategy, and (iii) write out a solution using Pólya's four-step process.

1. Consider the following products. Use your calculator to verify that the statements are true.

\[
1 \times (1) = 1^2 \\
121 \times (1 + 2 + 1) = 22^2 \\
12321 \times (1 + 2 + 3 + 2 + 1) = 333^2
\]

Predict the next line in the sequence of products. Use your calculator to check your answer.

2. **a.** How many cubes of all sizes are in a $2 \times 2 \times 2$ cube composed of eight $1 \times 1 \times 1$ cubes?

   **b.** How many cubes of all sizes are in an $8 \times 8 \times 8$ cube composed of 512 $1 \times 1 \times 1$ cubes?

3. **a.** What is the smallest number of whole-number gram weights needed to weigh any whole-number amount from 1 to 12 grams on a scale allowing the weights to be placed on either or both sides?

   **b.** How about from 1 to 37 grams?

   **c.** What is the most you can weigh using six weights in this way?

**CHAPTER TEST**

**KNOWLEDGE**

1. List the four steps of Pólya's problem-solving process.

2. List the six problem-solving strategies you have learned in this chapter.

**SKILL**

3. Identify the unneeded information in the following problem.

   Birgit took her $5 allowance to the bookstore to buy some back-to-school supplies. The pencils cost $0.10, the erasers cost $0.05 each, and the clips cost 2 for $0.01. If she bought 100 items altogether at a total cost of $1, how many of each item did she buy?

4. Rewrite the following problem in your own words.

   If you add the square of Ruben's age to the age of Angelita, the sum is 62; but if you add the square of Angelita's age to the age of Ruben, the sum is 176. Can you say what the ages of Ruben and Angelita are?

5. Given the following problem and its numerical answer, write the solution in a complete sentence.

   Amanda leaves with a basket of hard-boiled eggs to sell. At her first stop she sold half her eggs plus half an egg. At her second stop she sold half her eggs plus half an egg. The same thing occurs at her third, fourth, and fifth stops. When she finishes, she has no eggs in her basket. How many eggs did she start with?

   **Answer:** 31

**UNDERSTANDING**

6. Explain the difference between an exercise and a problem.

7. List at least two characteristics of a problem that would suggest using the Guess and Test strategy.
8. List at least two characteristics of a problem that would suggest using the Use a Variable strategy.

**PROBLEM SOLVING / APPLICATION**

For each of the following problems, read the problem carefully and solve it. Identify the strategy you used.

9. Can you rearrange the 16 numbers in this $4 \times 4$ array so that each row, each column, and each of the two diagonals total 10? How about a $2 \times 2$ array containing two 1s and two 2s? How about the corresponding $3 \times 3$ array?

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 \\
\end{array}
\]

10. In three years, Chad will be three times my present age. I will then be half as old as he. How old am I now?

11. There are six baseball teams in a tournament. The teams are lettered A through F. Each team plays each of the other teams twice. How many games are played altogether?

12. A fish is 30 inches long. The head is as long as the tail. If the head was twice as long and the tail was its present length, the body would be 18 inches long. How long is each portion of the fish?

13. The Orchard brothers always plant their apple trees in square arrays, like those illustrated. This year they planted 31 more apple trees in their square orchard than last year. If the orchard is still square, how many apple trees are there in the orchard this year?

\[
\begin{array}{c}
\bullet \bullet \bullet \\
\circ \circ \circ \\
1 \quad 4 \quad 9
\end{array}
\]

14. Arrange 10 people so that there are five rows each containing 4 persons.

15. A milk crate holds 24 bottles and is shaped like the one shown below. The crate has four rows and six columns. Is it possible to put 18 bottles of milk in the crate so that each row and each column of the crate has an even number of bottles in it? If so, how? (Hint: One row has 6 bottles in it and the other three rows have 4 bottles in them.)

16. Otis has 12 coins in his pocket worth $1.10. If he only has nickels, dimes, and quarters, what are all of the possible coin combinations?

17. Show why 3 always divides evenly into the sum of any three consecutive whole numbers.

18. If 14 toothpicks are arranged to form a triangle so none of the toothpicks are broken or bent and all 14 toothpicks are used, how many different-shaped triangles can be formed?

19. Together a baseball and a football weigh 1.25 pounds, the baseball and a soccer ball weigh 1.35 pounds, and the football and the soccer ball weigh 1.6 pounds. How much does each of the balls weigh? Explain your reasoning.

20. In the figure below, there are 7 chairs arranged around 5 tables. How many chairs could be placed around a similar arrangement of 31 triangular tables?

21. Carlos’ father pays Carlos his allowance each week using patterns. He pays a different amount each day according to some pattern. Carlos must identify the pattern in order to receive his allowance. Help Carlos complete the pattern for the missing days in each week below.

   a. 5¢, 9¢, 16¢, 26¢, _____, _____, _____
   b. 1¢, 6¢, 15¢, 30¢, 53¢, _____, _____
   c. 4¢, 8¢, 16¢, 28¢, _____, _____, _____